CS 360: Machine Learning

Prof. Sara Mathieson

Fall 2019

HAVERFORD
COLLEGE

Outline for October 31

- Reading Quiz
- Recap Perceptron Algorithm
- Introduction to Support Vector Machines
- Lab check in TODAY! (Parts 1&2 complete)

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Reading Quiz

1. What is the goal of the perceptron algorithm? Circle all that apply:

   (a) predict a continuous outcome

   (b) quantify how important each feature is for predicting the outcome

   (c) create a linear decision boundary between positives and negatives

   (d) obtain the probability of a positive label for each test example

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   True

3. Say at some point in the perceptron algorithm I have $\vec{w} = [3, -1, 2]^T$ and $\vec{x} = [1, 2, -2]^T$. What label would we predict for $\vec{x}$?

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Dot product = -3 => predict label -1

4. In the example above, say the true label is -1. How would the weights be updated when using this point?

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   No weight update!

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Perceptron as a neural network

Biological model of a neuron

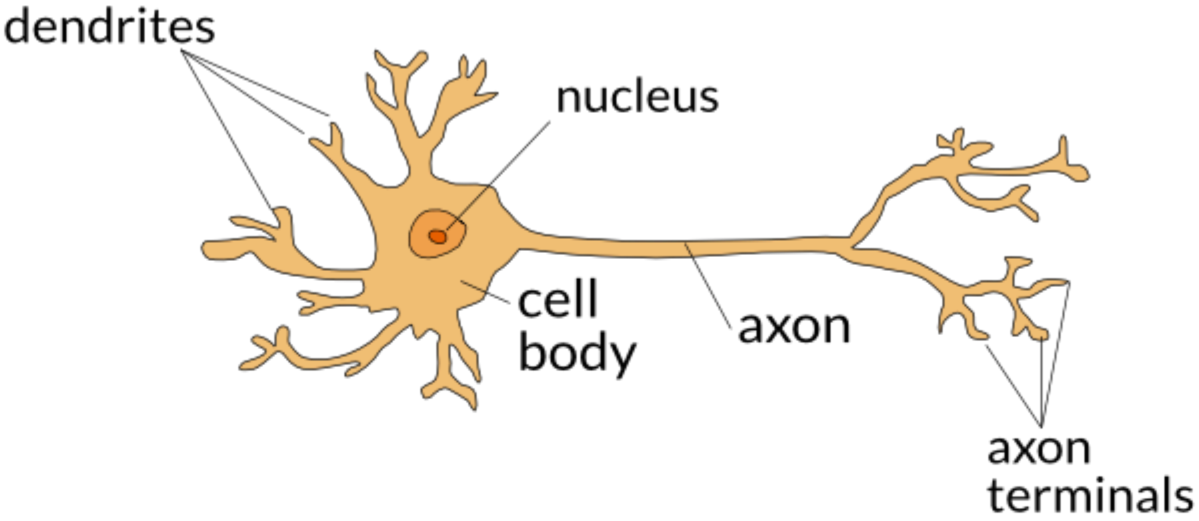


Image: modified from "Applied Go"

Perceptron as a neural network

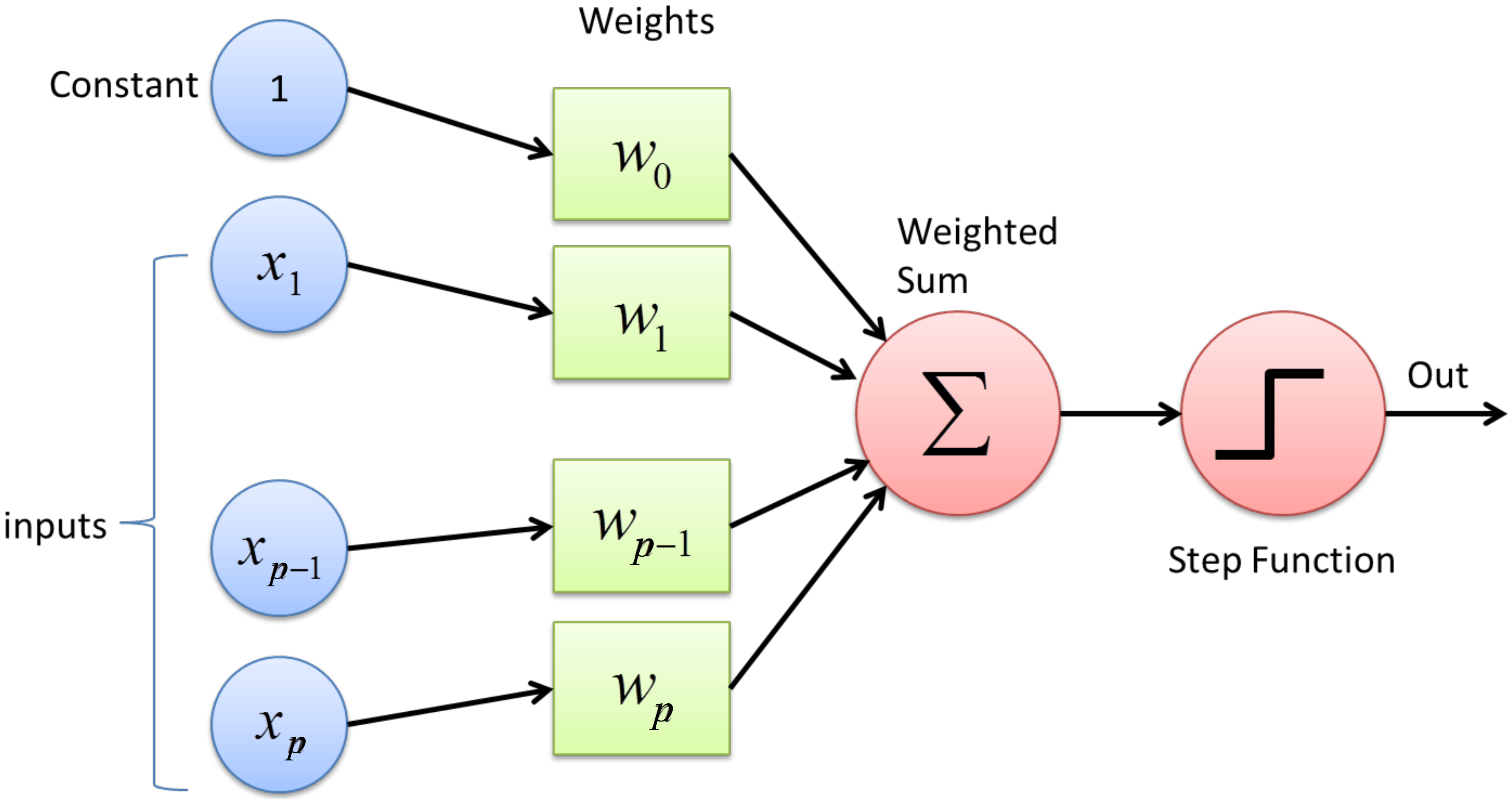


Image: modified from “Towards Data Science”

History of the Perceptron

- Invented in 1957 by Frank Rosenblatt
- Initially thought to be the "solution to AI"

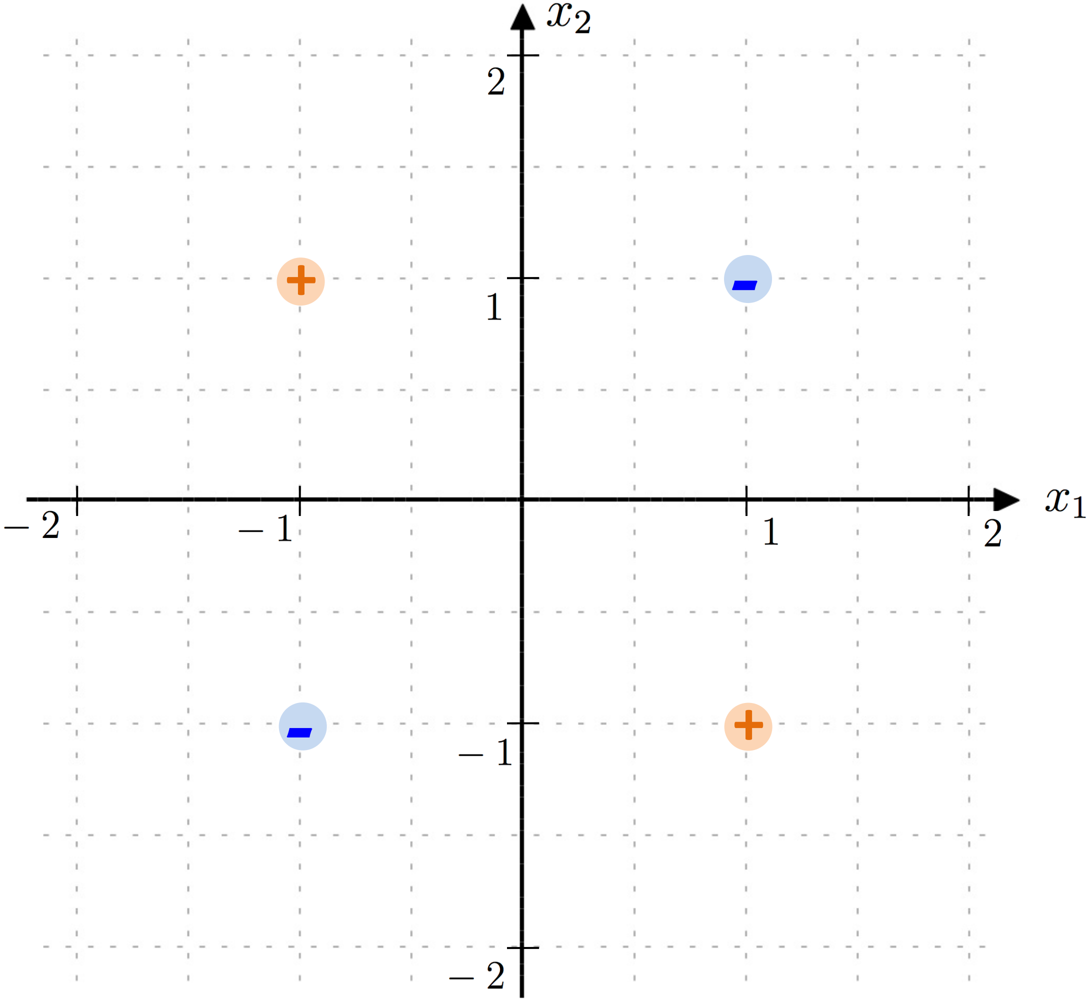
  NYT said the perceptron was "*the embryo of an electronic computer that [the Navy] expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence*"

- Famous book "Perceptrons" by Marvin Minsky and Seymour Papert (1969)
- Confusion about the text contributed to first "AI winter"

Perceptron cannot learn XOR

($x_1 = 1$ or $x_2 = 1$, but not both)

Why?



Perceptron cannot learn XOR

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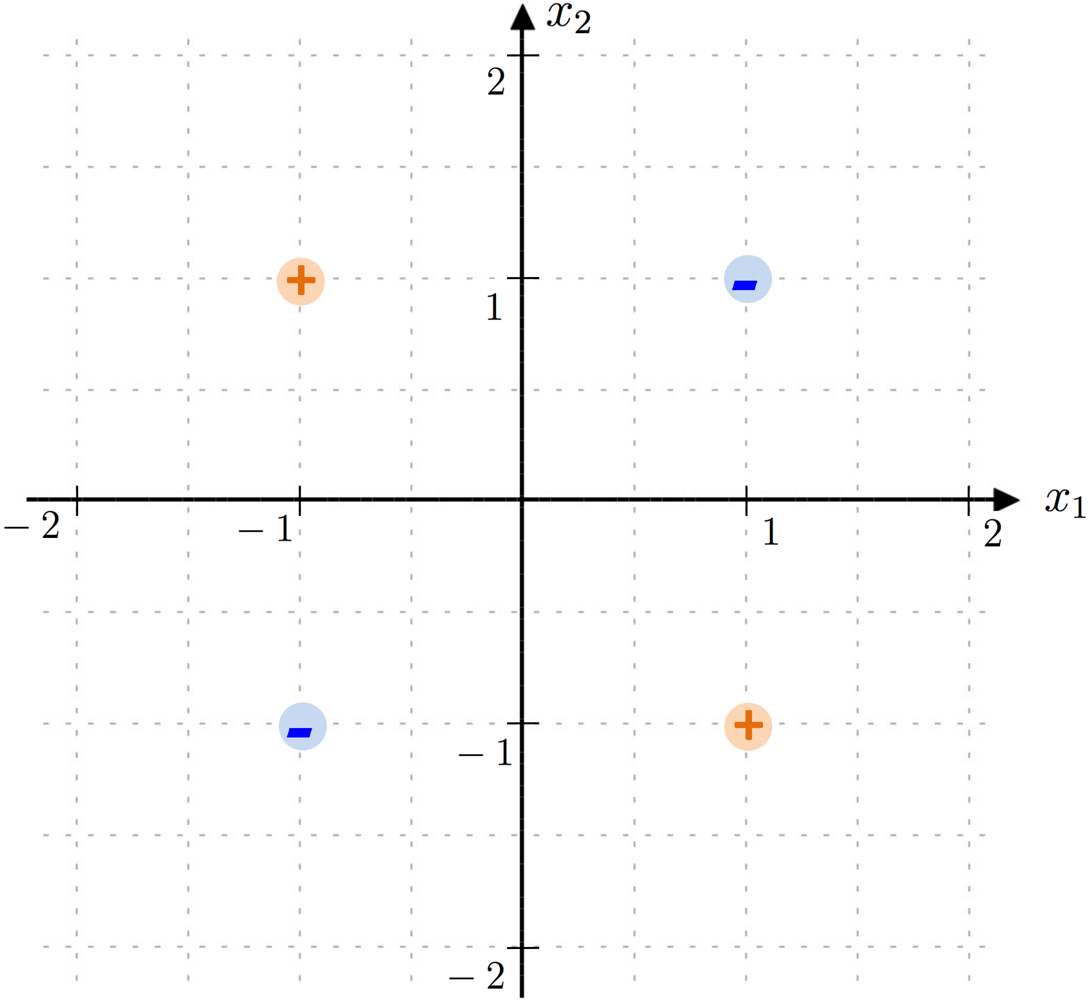
Why?

Not linearly separable!



Convergence Guarantee

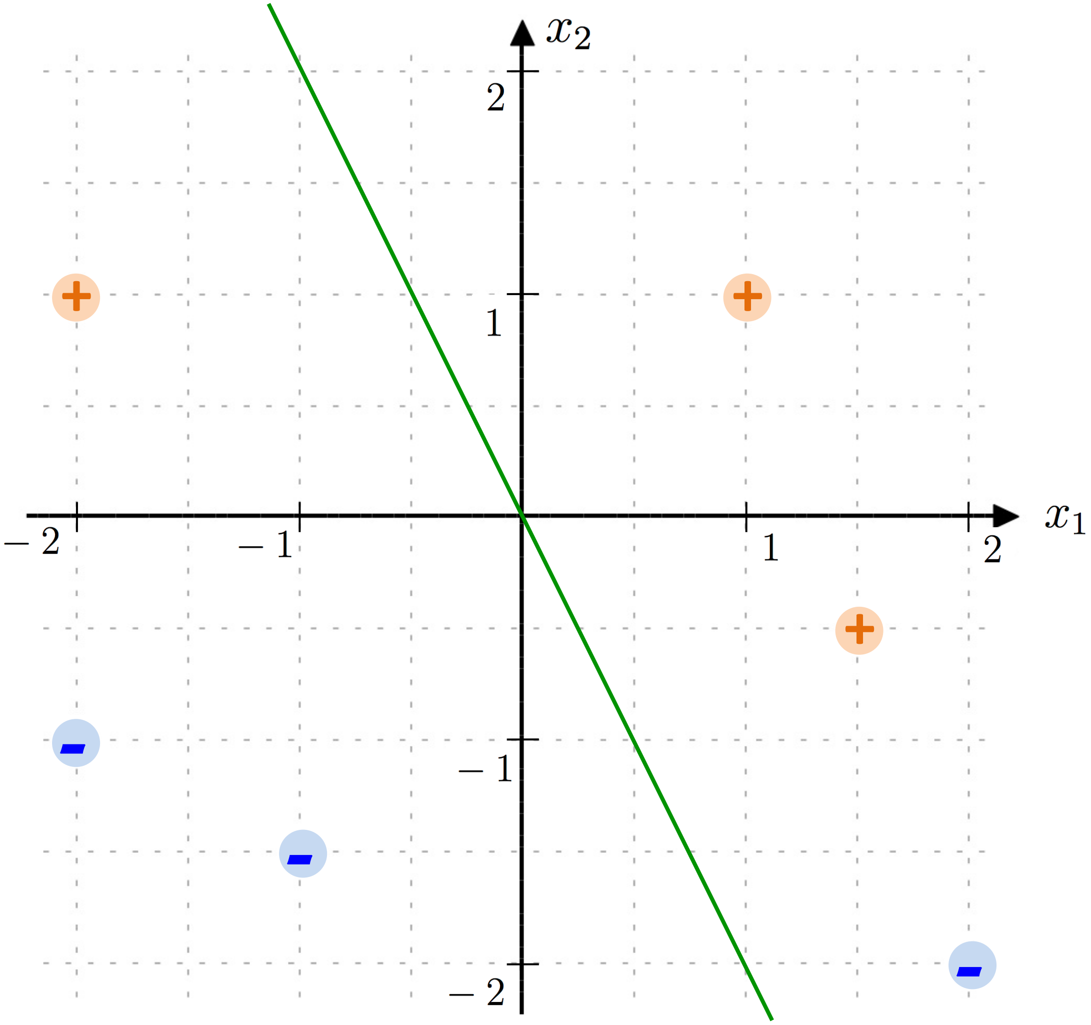
- Perceptron is guaranteed to converge to a solution if a separating hyperplane exists
- Not guaranteed to converge to a “good” solution
- No guarantees about behavior if a separating hyperplane does not exist!

Handout 15 example

Initial values:

$$\alpha = 0.2$$

$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 15 example

$$\alpha = 0.2$$

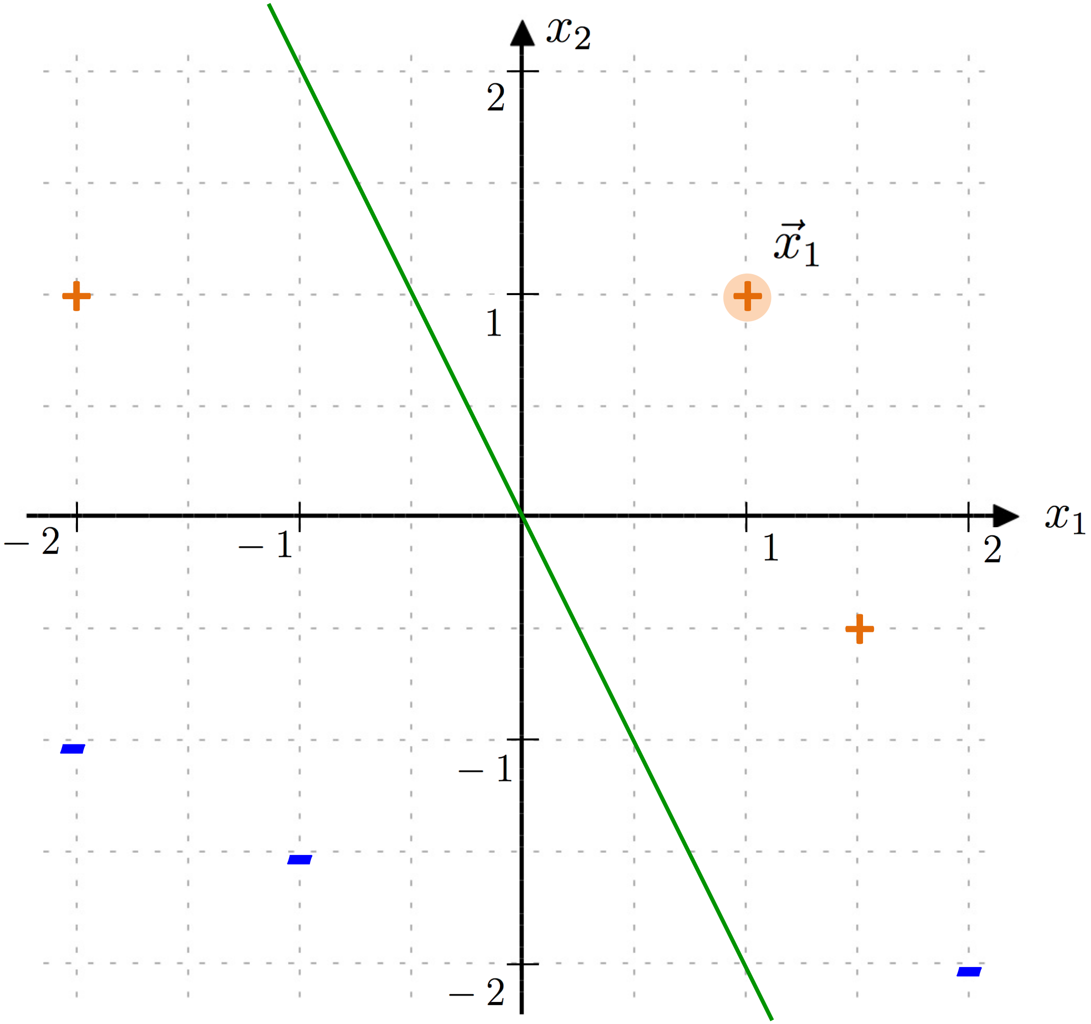
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 1:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_1 > 0$$

Correct classification, no action



Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 15 example

$$\alpha = 0.2$$

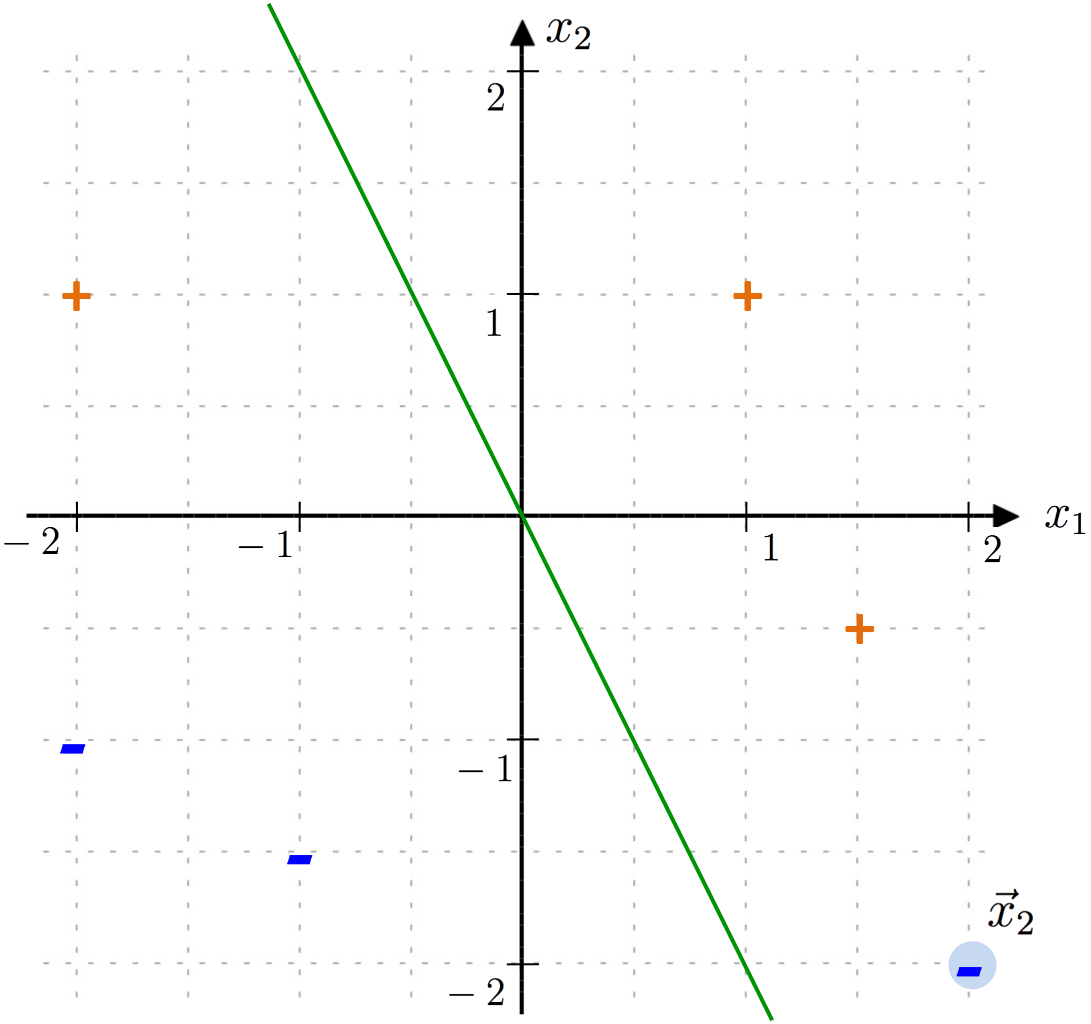
$$\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}$$

Round 2:

$$\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w} \cdot \vec{x}_2 > 0$$

Incorrect classification



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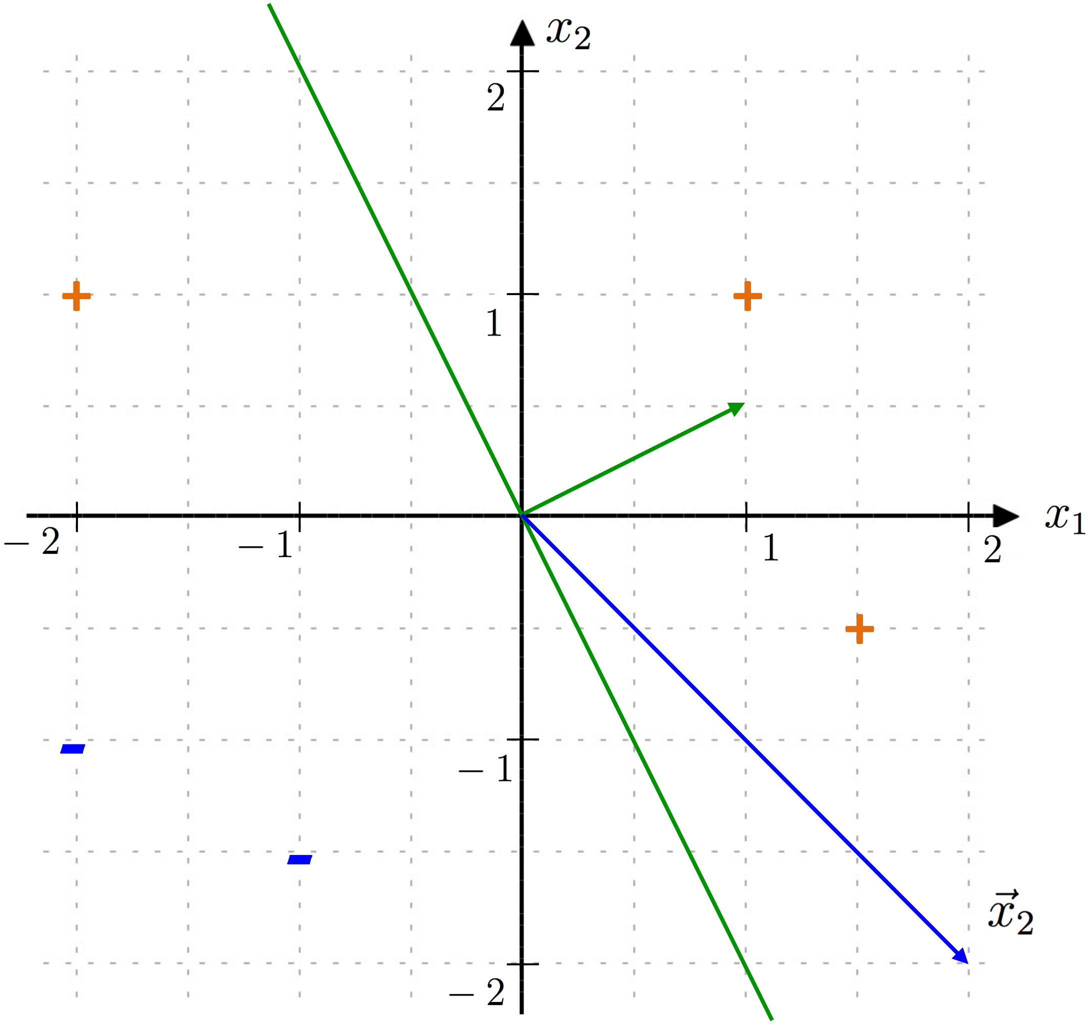

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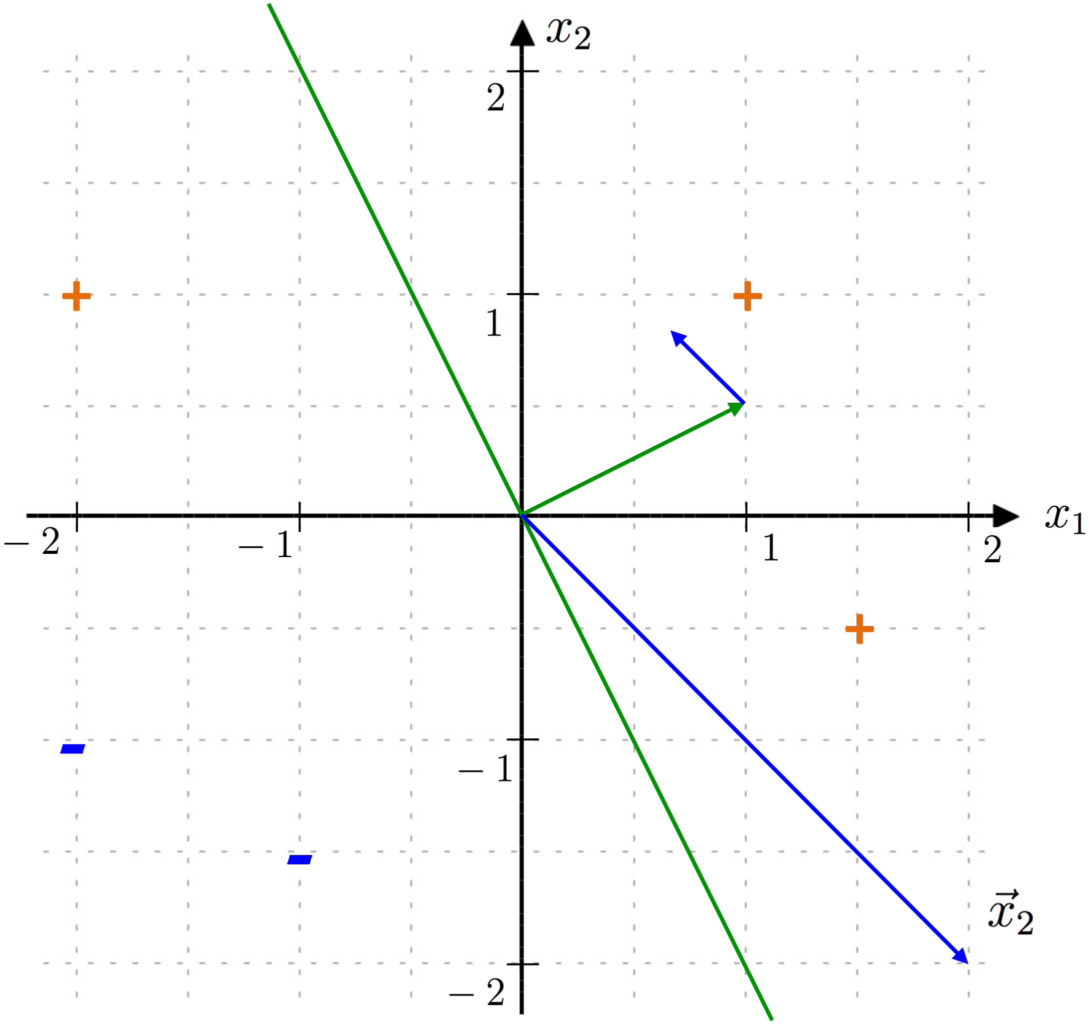
Incorrect classification
"Push" ***w*** away from negative point



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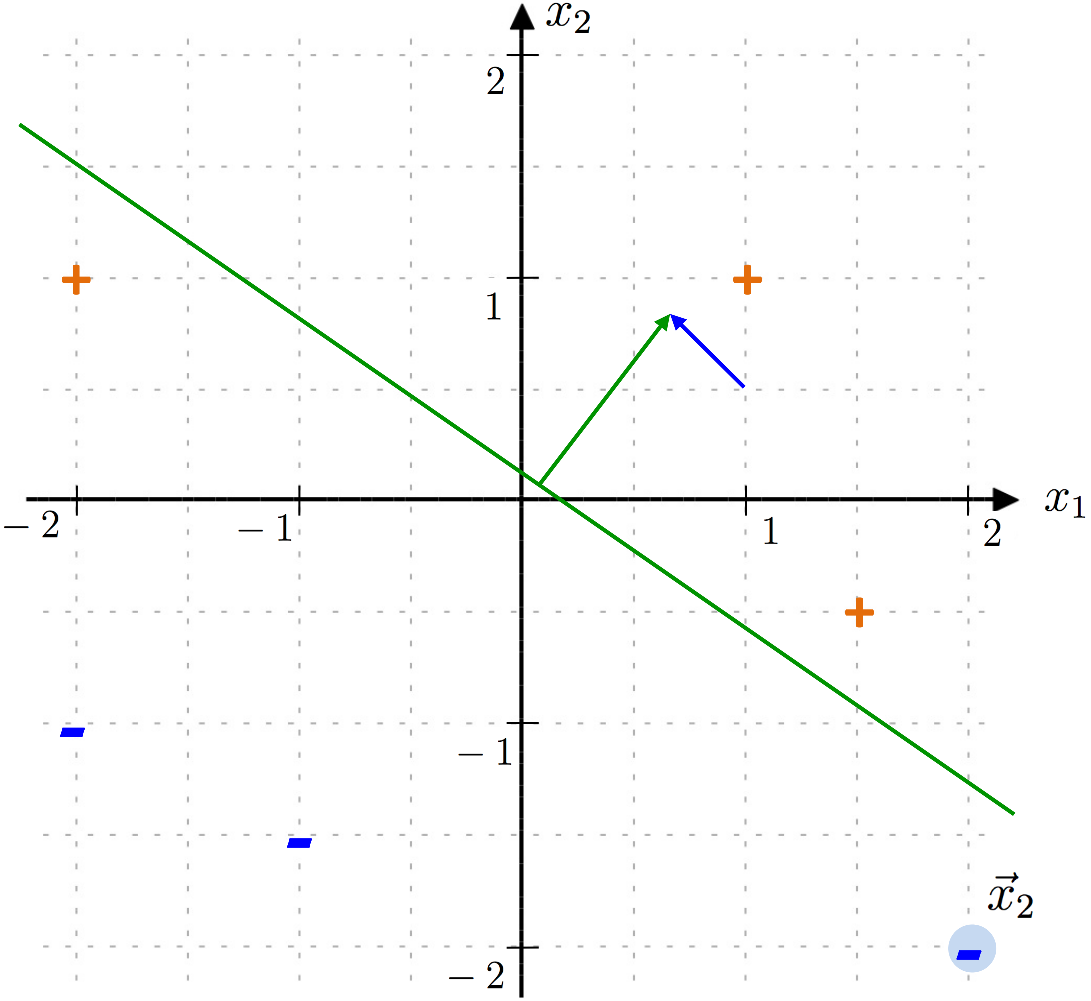


Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Handout 15 example

Final solution (so you can check your work):

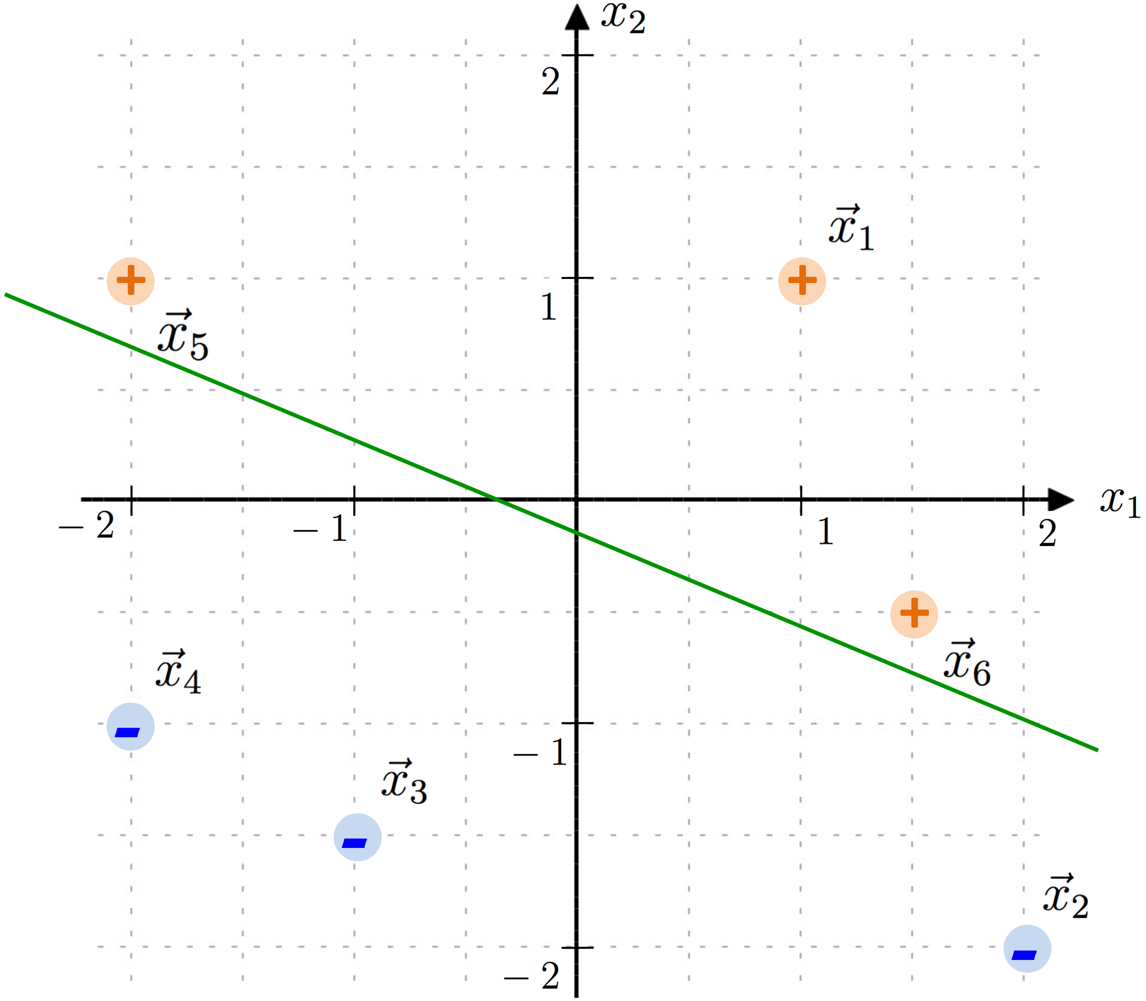
$$\vec{w}^* = \begin{bmatrix} 0.2 \\ 0.5 \\ 1 \end{bmatrix}$$

Final hyperplane:

$$0.2 + 0.5x_1 + x_2 = 0$$

$$\Rightarrow$$

$$x_2 = -0.2 - 0.5x_1$$



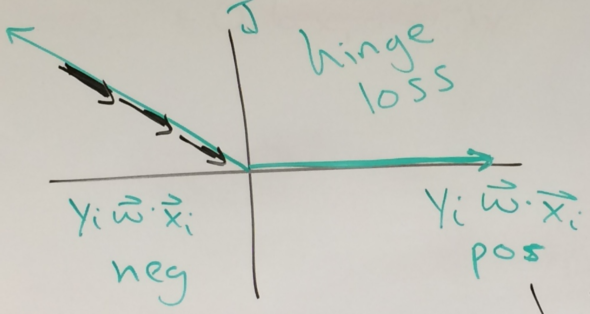
Example modified from Achim J. Lilienthal & Thorsteinn Rögnvaldsson

Perceptron Updates

in terms of cost.

$$J(\vec{w}) = \sum_{i=1}^{n} \max(0, \underbrace{-y_i \vec{w}\cdot\vec{x}_i})$$

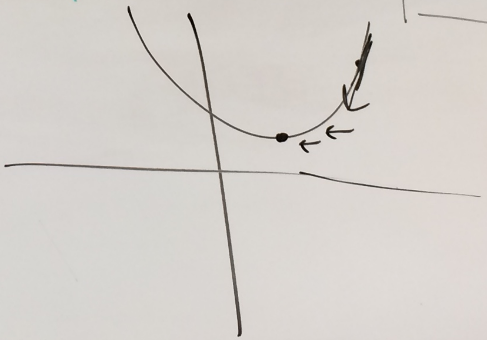
if same sign then max is 0. (correct)



$$\boxed{\nabla J(\vec{w}) = -y_i \vec{x}_i}$$

updates. SGD

$$\boxed{\vec{w} \leftarrow \vec{w} + \alpha y_i \vec{x}_i}$$



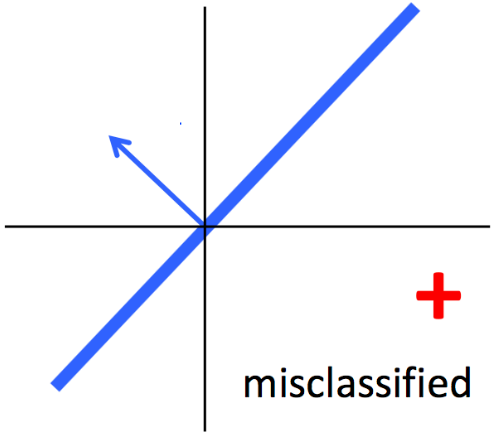
Informal discussion with a partner

1) What is the relationship between the weight vector ***w*** and the hyperplane?

2) Why is the perceptron cost function intuitive?

$$J(\vec{w}) = \sum_{i=1}^{n} \max\left(0, -y_i(\vec{w}^T \vec{x}_i)\right)$$

3) In the example to the right, how will the slope of the hyperplane change?



4) What are the weaknesses of the perceptron? Create a binary classifier "wishlist".

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They are perpendicular

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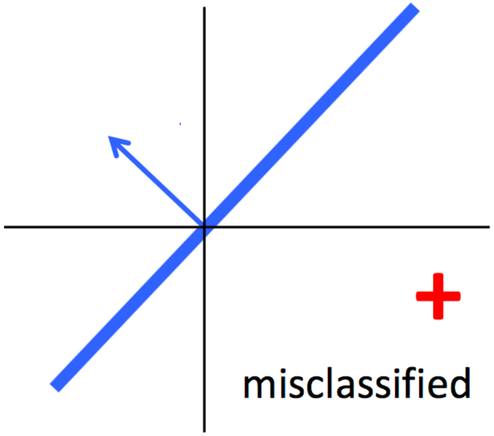

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Cost function is 0 when classification is correct, and positive when incorrect

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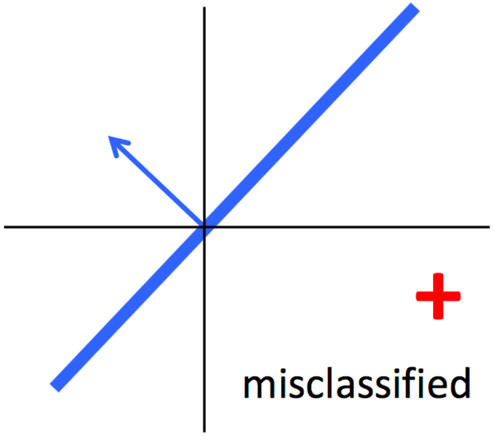

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Perceptron algorithm and intuition

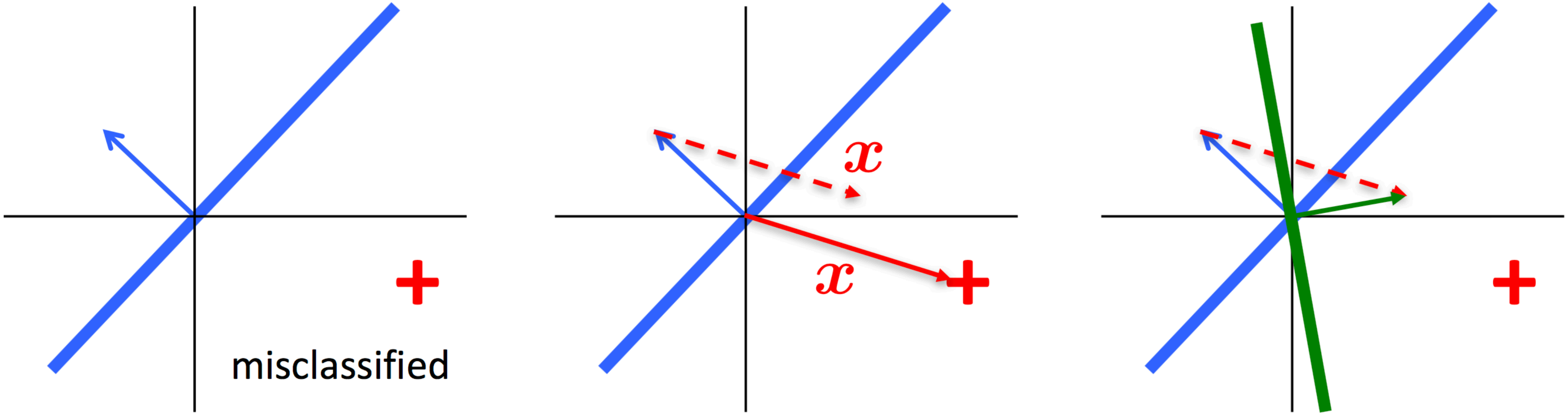
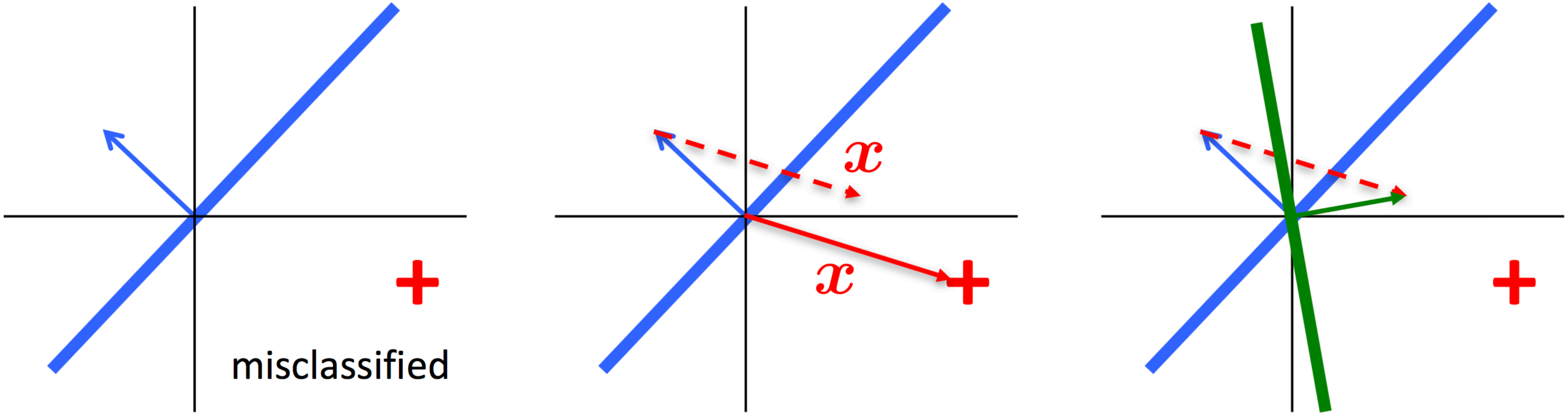


Image and Algorithm: modified from Eric Eaton

Perceptron algorithm and intuition



```
Let $\vec{w} = [0, 0, \cdots, 0]^T$
Repeat until convergence:
    Receive training example $(\vec{x}_i, y_i)$
    If $y_i(\vec{w}^T \vec{x}_i) \leq 0$    (incorrectly classified)
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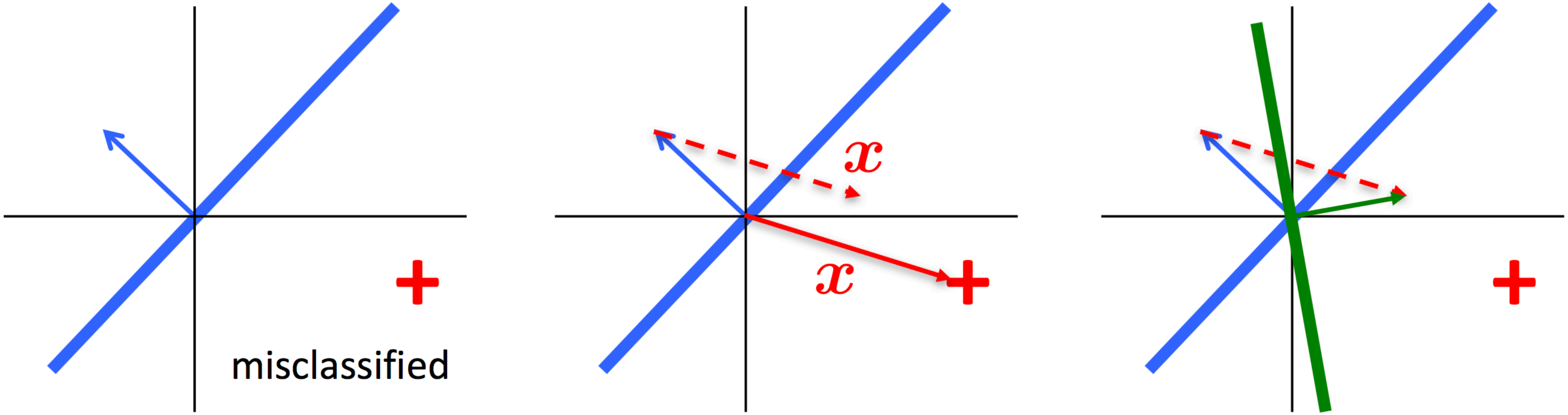

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```

Convergence:

- All data points correctly classified
- Fixed number of iterations passed

Often: alpha = 1 (only changes magnitude of weight vector)

Image and Algorithm: modified from Eric Eaton

Binary classifier wishlist

- If data is linearly separable, want a “good” hyperplane (idea: far from points close to the boundary)

- If data is not linearly separable, want something reasonable (not just give up or fail to converge)

- Might not want to constrain ourselves to linear separators

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Support Vector Machines (SVMs)

- Will give us everything on our wishlist!
- Often considered the best “off the shelf” binary classifier
- Widely used in many fields

Brief history

- 1963: Initial idea by Vladimir Vapnik and Alexey Chervonenkis
- 1992: nonlinear SVMs by Bernhard Boser, Isabelle Guyon and Vladimir Vapnik
- 1993: “soft-margin” by Corinna Cortes and Vladimir Vapnik

Idea: “best” hyperplane has a large margin

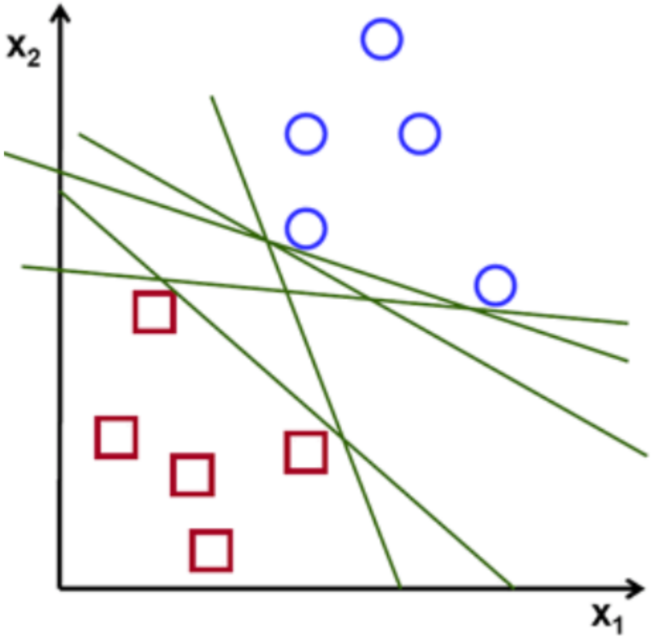

Image: “Towards Data Science”

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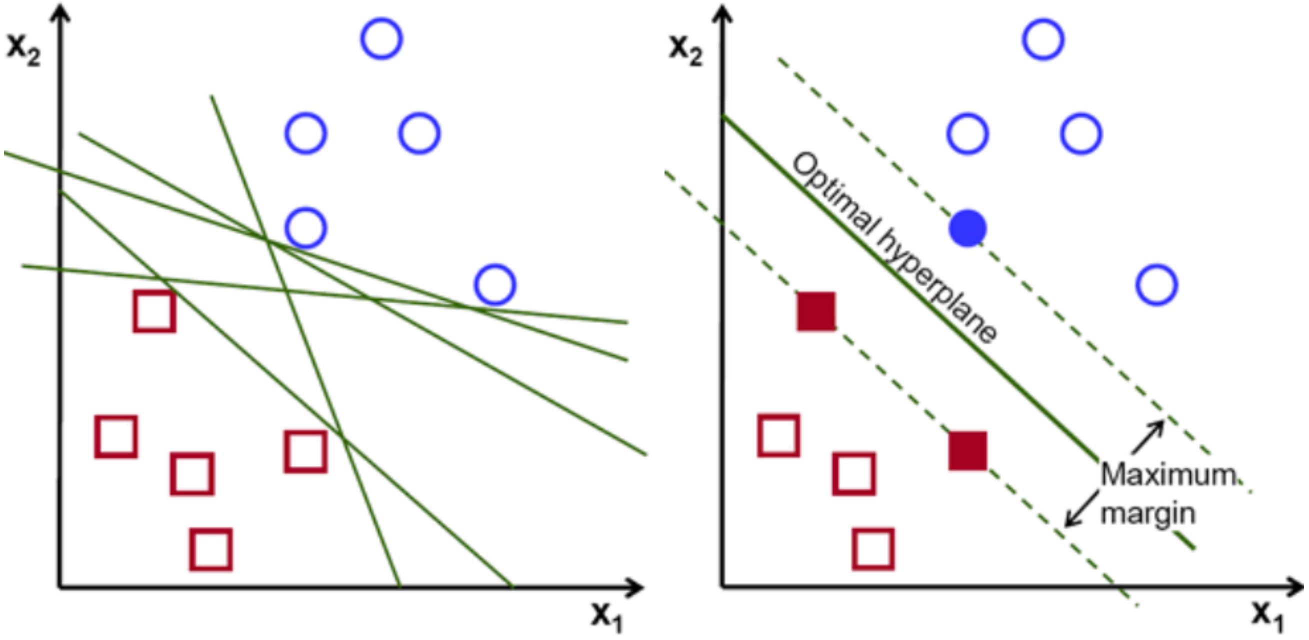

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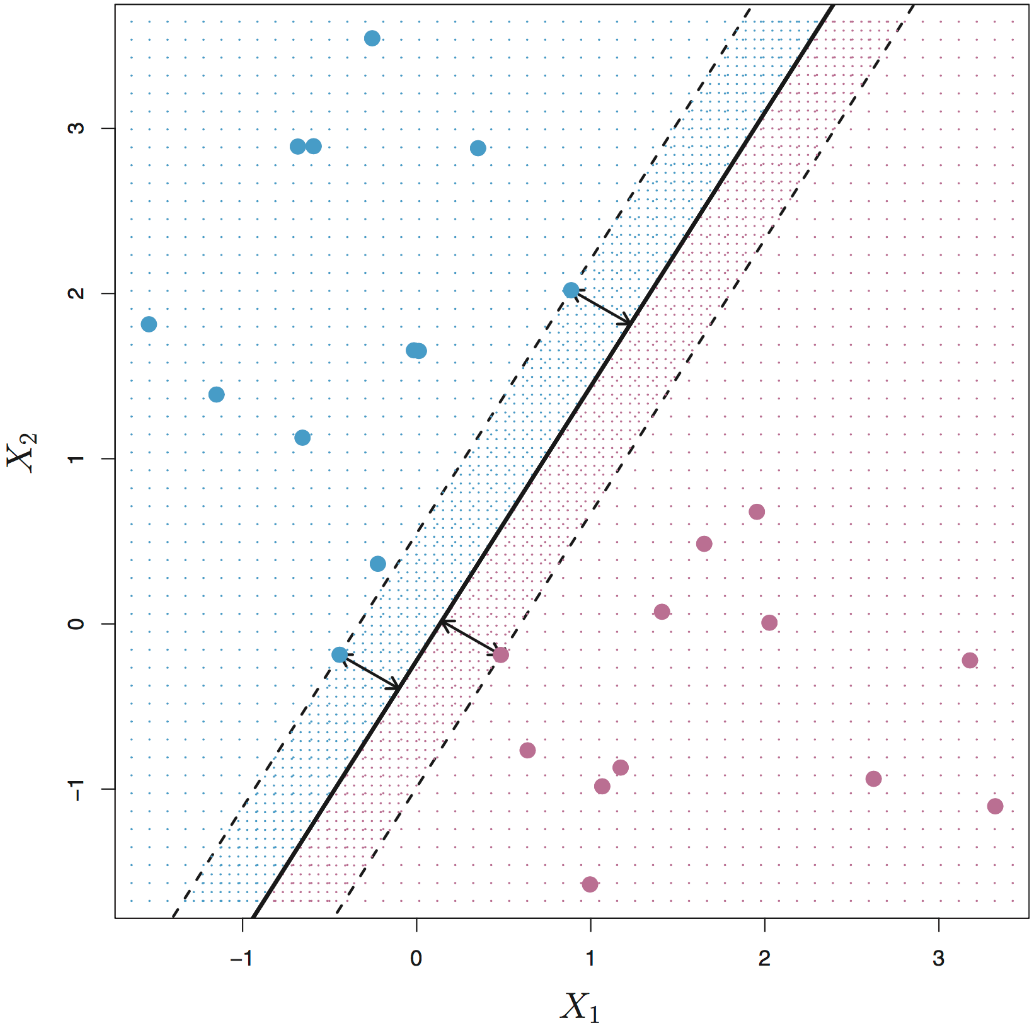
Datapoints that lie on the margin are called "support vectors"



Modified from ISL: Figure 9.3

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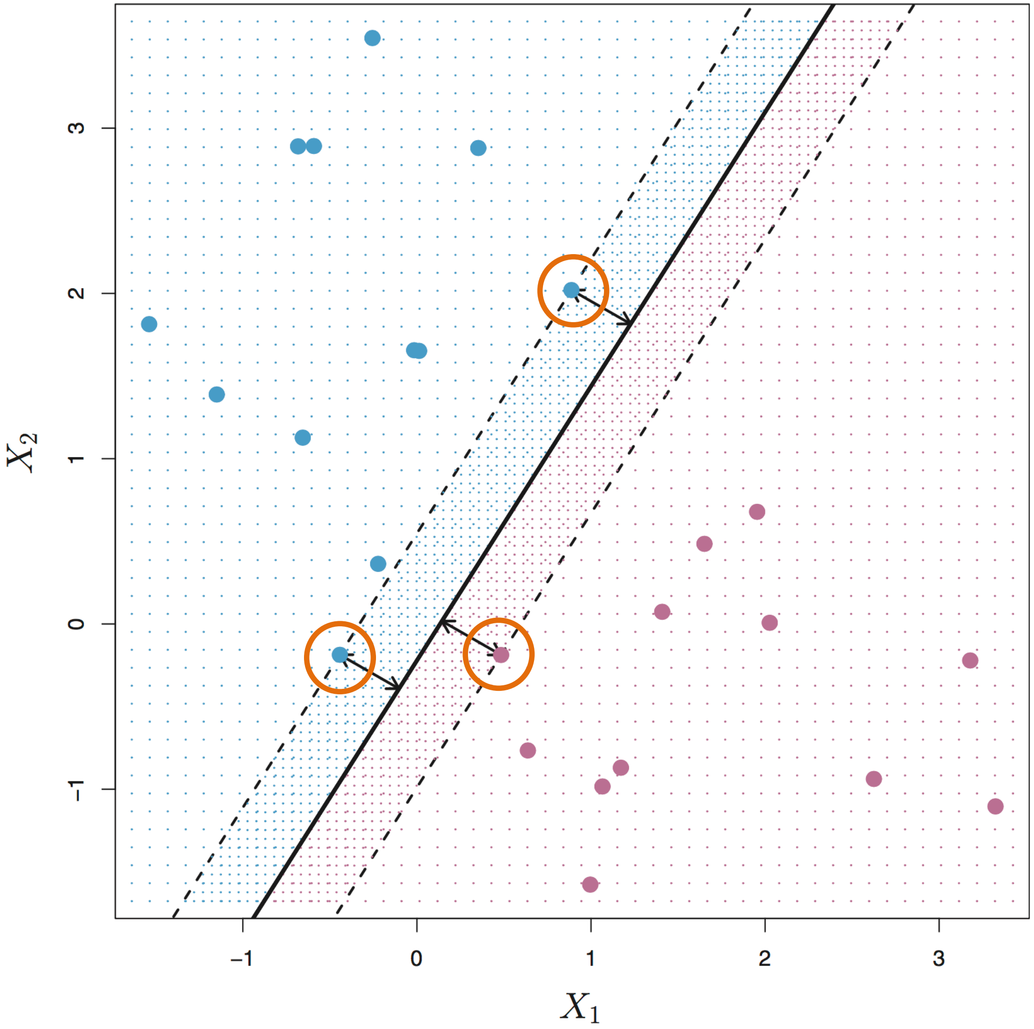


Support vectors

Modified from ISL: Figure 9.3

Support Vector Machines

let $h(\vec{x}) = \text{sign}(\vec{w}\cdot\vec{x}+b)$ same as perceptron

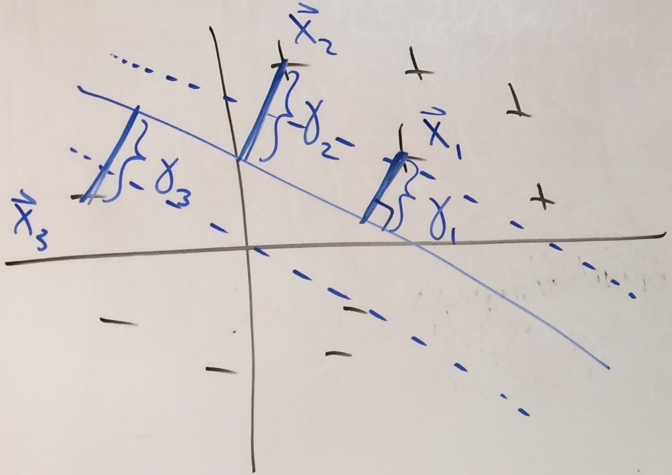
functional margin $\boxed{\hat{\gamma}_i = y_i(\vec{w}\cdot\vec{x}_i + b)}$

if correct: $\hat{\gamma}_i > 0$ ☆

if incorrect: $\hat{\gamma}_i < 0$ ✗

bad: increase magnitude of $\vec{w}$ & and b to increase $\hat{\gamma}_i$

good: arbitrary constraint on $\vec{w}$ & b.



Idea: want to maximize $\boxed{\hat{\gamma} = \min_{i=1\cdots n} \hat{\gamma}_i}$

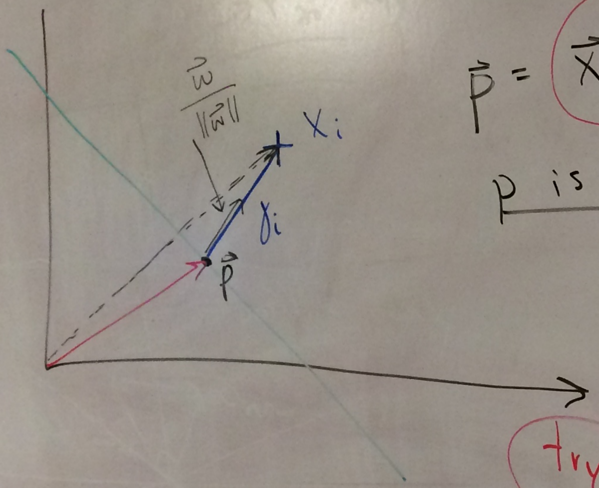
overall functional margin

Geometric margin → $\gamma_i = ?$

distance between pt & hyperplane

$$\boxed{\vec{p} + \gamma_i y_i \frac{\vec{w}}{\|\vec{w}\|} = \vec{x}_i}$$

unit vector



$$\vec{p} = \vec{x}_i - \gamma_i y_i \frac{\vec{w}}{\|\vec{w}\|}$$

p is on the hyperplane

$$0 = \vec{w} \cdot \vec{p} + b$$

plug in $\vec{p}$ &
solve for γ_i

exercise!

$$\boxed{\gamma_i = y_i \left(\frac{\vec{w}}{\|\vec{w}\|} \cdot \vec{x}_i + \frac{b}{\|\vec{w}\|} \right)}$$

try to maximize!!

Functional and Geometric Margins

SVM classifier:

(same as perceptron)

$$h(\vec{x}) = \operatorname{sign}\big(\vec{w} \cdot \vec{x} + b\big)$$

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Functional Margin:

$$\hat{\gamma}_i = y_i(\vec{w} \cdot \vec{x}_i + b)$$

Functional and Geometric Margins

SVM classifier:

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Functional Margin:

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Geometric Margin:

(distance between example and hyperplane)

$$\gamma_i = y_i \left(\frac{\vec{w}}{||\vec{w}||} \cdot \vec{x}_i + \frac{b}{||\vec{w}||} \right)$$

Functional and Geometric Margins

SVM classifier:
(same as perceptron)

$$h(\vec{x}) = \operatorname{sign}\left(\vec{w} \cdot \vec{x} + b\right)$$

Functional Margin:

$$\hat{\gamma}_i = y_i(\vec{w} \cdot \vec{x}_i + b)$$

Geometric Margin:
(distance between
example and hyperplane)

$$\gamma_i = y_i \left(\frac{\vec{w}}{||\vec{w}||} \cdot \vec{x}_i + \frac{b}{||\vec{w}||} \right)$$

Note:

$$\boxed{\gamma_i = \frac{\hat{\gamma}_i}{||\vec{w}||}}$$

Optimization Problem: try 1

Goal: maximize the minimum distance between example and hyperplane

$$\gamma = \min_{i=1,\cdots,n} \gamma_i$$

Optimization Problem: try 1

Goal: maximize the minimum distance between example and hyperplane

$$\gamma = \min_{i=1,\cdots,n} \gamma_i$$

Formulation: optimize a function with respect to a constraint

$$\begin{aligned} \max_{\gamma,\vec{w},b} \quad & \gamma \\ \text{s.t.} \quad & y_i(\vec{w} \cdot \vec{x}_i + b) \geq \gamma, \quad i = 1, \cdots, n \\ \text{and} \quad & \|\vec{w}\| = 1 \end{aligned}$$

(force functional and geometric margin to be equal)

Optimization Problem: try 2

Idea: substitute functional margin
divided by magnitude of weight vector

$$\max_{\hat{\gamma},\vec{w},b} \quad \frac{\hat{\gamma}}{\|\vec{w}\|}$$

$$\text{s.t.} \quad y_i(\vec{w}\cdot\vec{x}_i + b) \geq \hat{\gamma}, \quad i = 1,\cdots,n$$

(gets rid of non-convex constraint)

Optimization Problem: try 3

Idea: put arbitrary constraint on functional margin

$$\boxed{\hat{\gamma} = 1}$$

$$\begin{aligned} \min_{\vec{w},b} \quad & \frac{1}{2}\|\vec{w}\|^2 \\ \text{s.t.} \quad & y_i(\vec{w}\cdot\vec{x}_i + b) \geq 1, \quad i = 1,\cdots,n \end{aligned}$$

Optimization Problem: try 3

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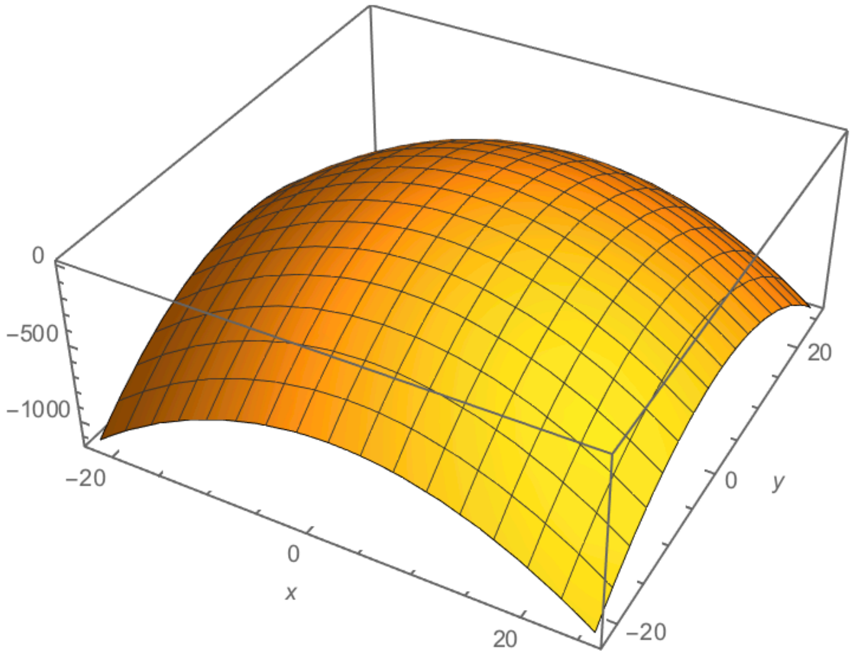
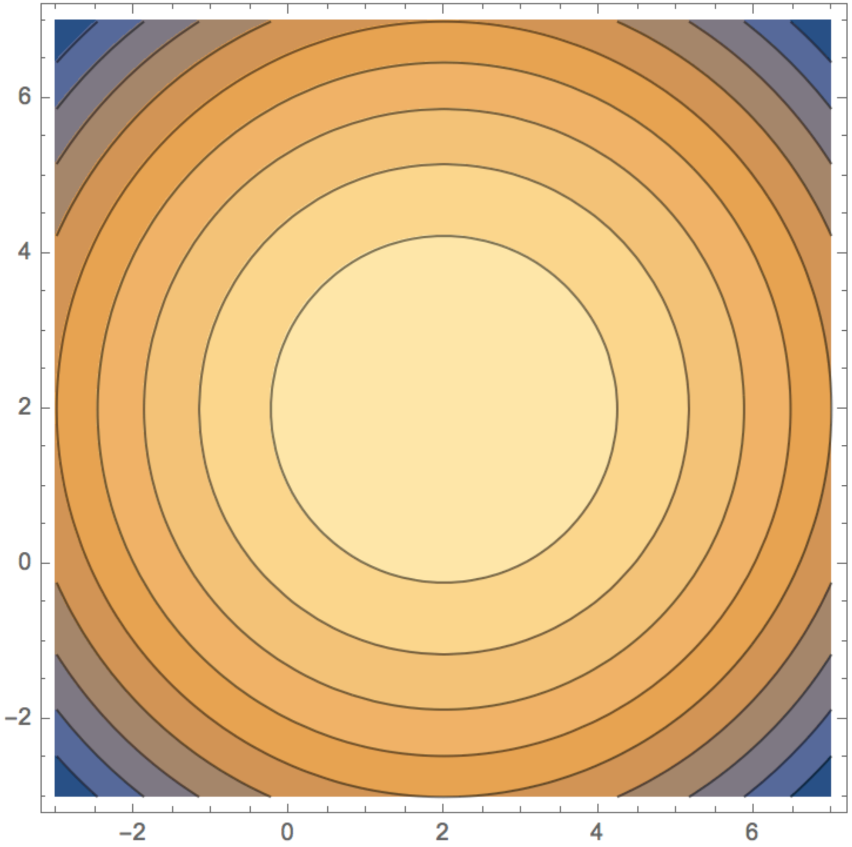
$$\begin{aligned} \min_{\vec{w},b} \quad & \frac{1}{2}\|\vec{w}\|^2 \\ \text{s.t.} \quad & -y_i(\vec{w}\cdot\vec{x}_i + b) + 1 \leq 0, \quad i = 1, \cdots, n \end{aligned}$$

Lagrange multipliers example 1

$$f(x,y) = 5 - (x-2)^2 - (y-2)^2$$





Contour plot of *f*(*x*,*y*)

$$\text{maximize}_{x,y} \quad f(x,y)$$

$$s.t. \quad g(x,y) = 0$$

$$g(x,y) = -5 + x + y$$

Detour to Lagrange Multipliers

Goal: * maximize function subject to constraint

$$\underset{x,y}{\text{maximize}} \quad f(x,y)$$

$$\text{s.t.} \quad g(x,y) = 0$$

no constraint $\Rightarrow x^* = 2$, $y^* = 2$

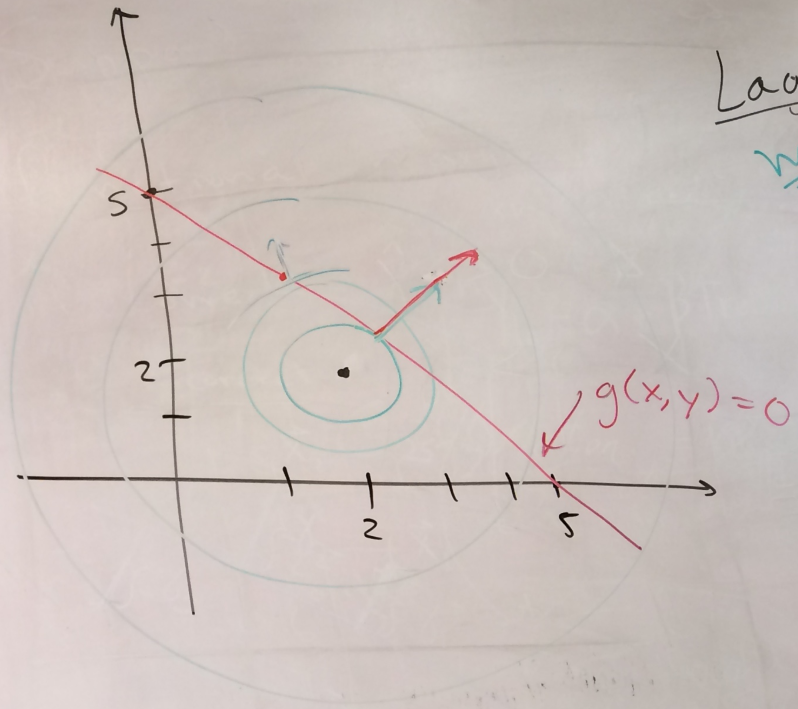
example: $\overset{\text{max}}{f(x,y)} = 5 - (x-2)^2 - (y-2)^2$

$$\text{s.t.} \quad g(x,y) = -5 + x + y$$

$$\boxed{g(x,y) = 0}$$

$$-5 + x + y = 0$$

$$\Rightarrow \boxed{y = -x + 5}$$



Lagrangian

could also add.

max $h(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$

Lagrange multiplier

Lagrange multiplier

$\nabla h(x,y,\lambda) = 0 \}$ want!

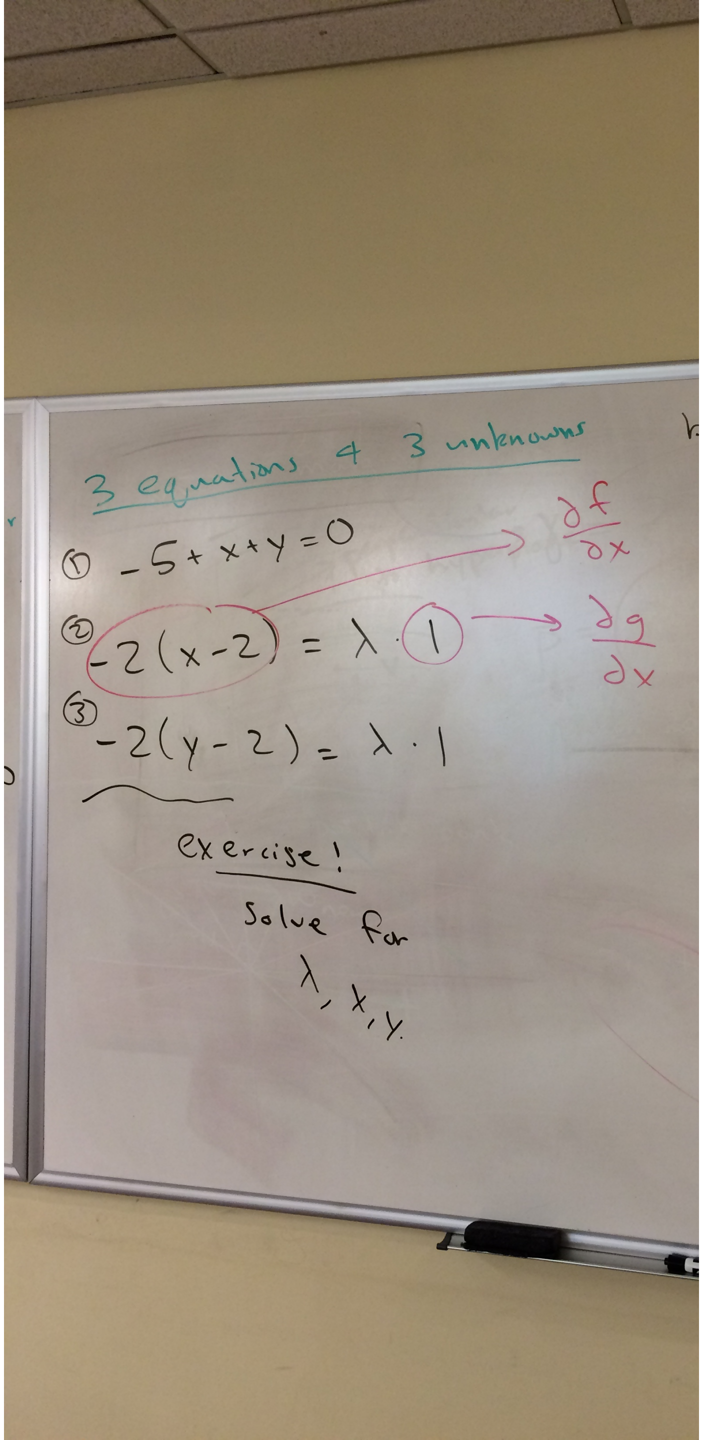
$\nabla_{x,y} h(x,y,\lambda) = \nabla f(x,y) - \lambda \nabla g(x,y) = 0$

$\frac{\partial h}{\partial \lambda} = \boxed{g(x,y) = 0}$ 1 equation

constraint $\Rightarrow \boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$

2 equations

①
②
③



3 equations & 3 unknowns

① $-5+x+y=0$

② $\underbrace{-2(x-2)}_{\frac{\partial f}{\partial x}} = \lambda \cdot \underbrace{1}_{\frac{\partial g}{\partial x}}$

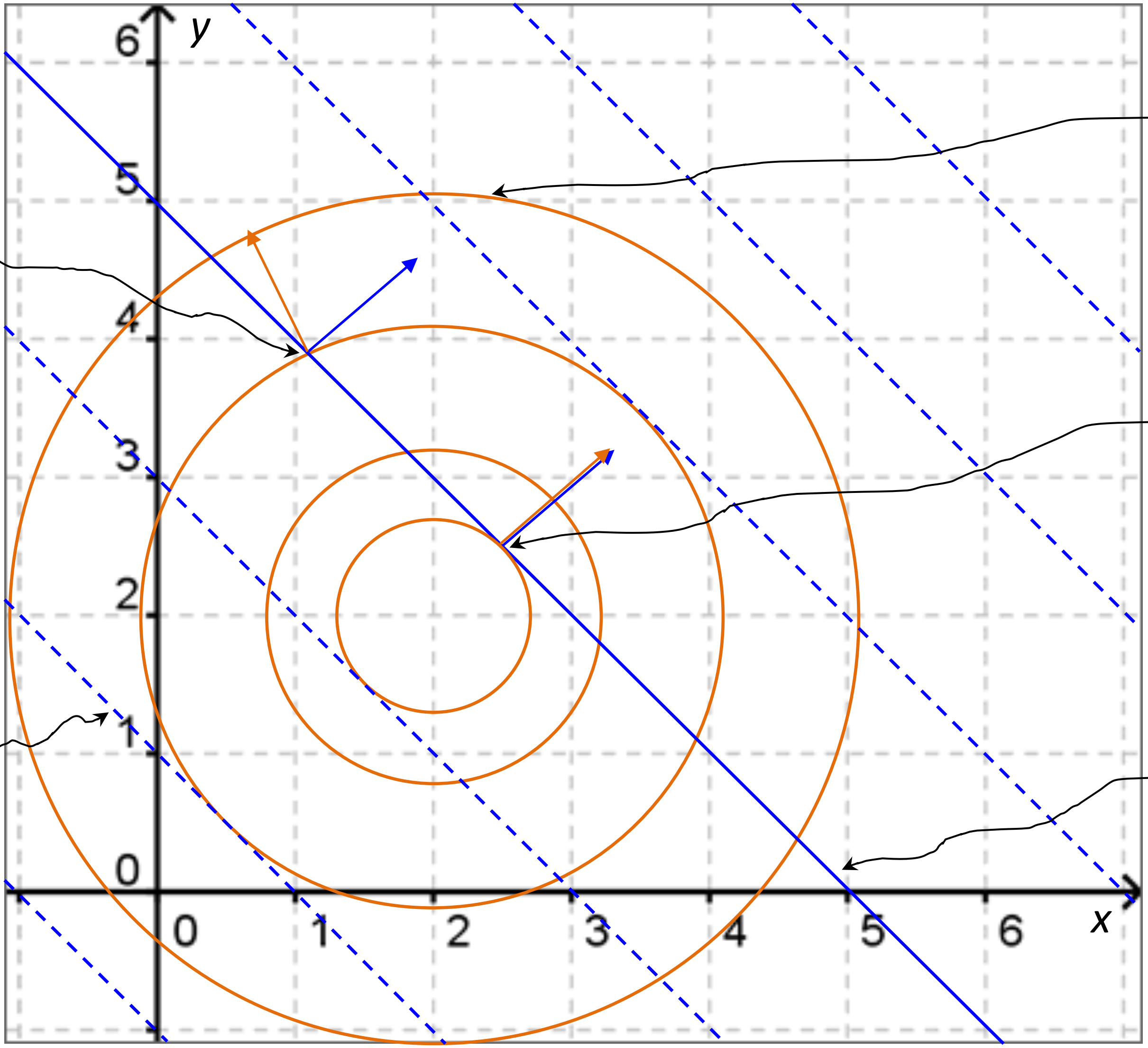
③ $-2(y-2) = \lambda \cdot 1$

exercise!

Solve for

$\lambda, x, y.$

Lagrange multipliers example 1



Normals (and derivatives) **not** parallel

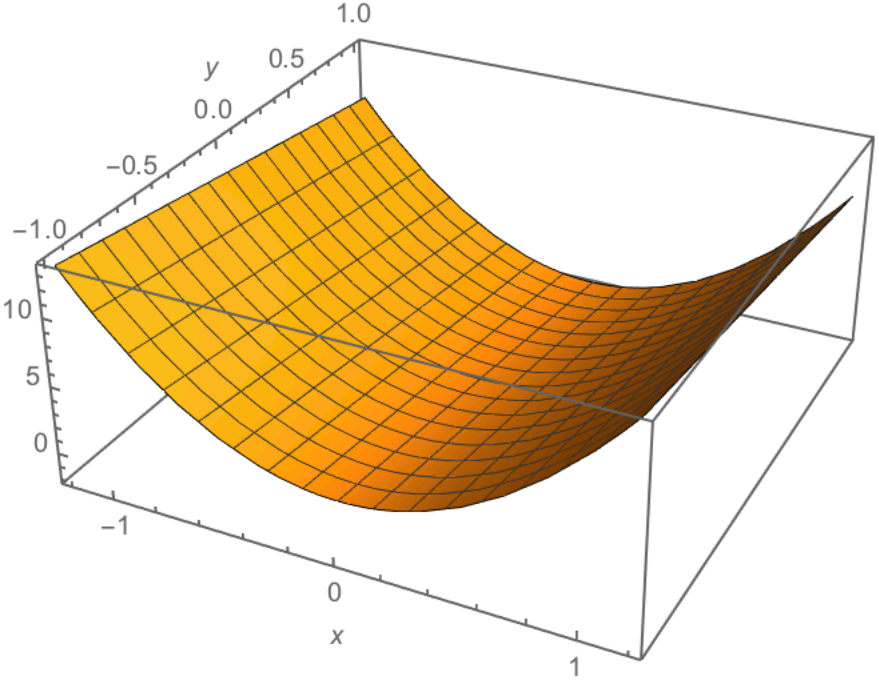
level curves of $g(x,y)$

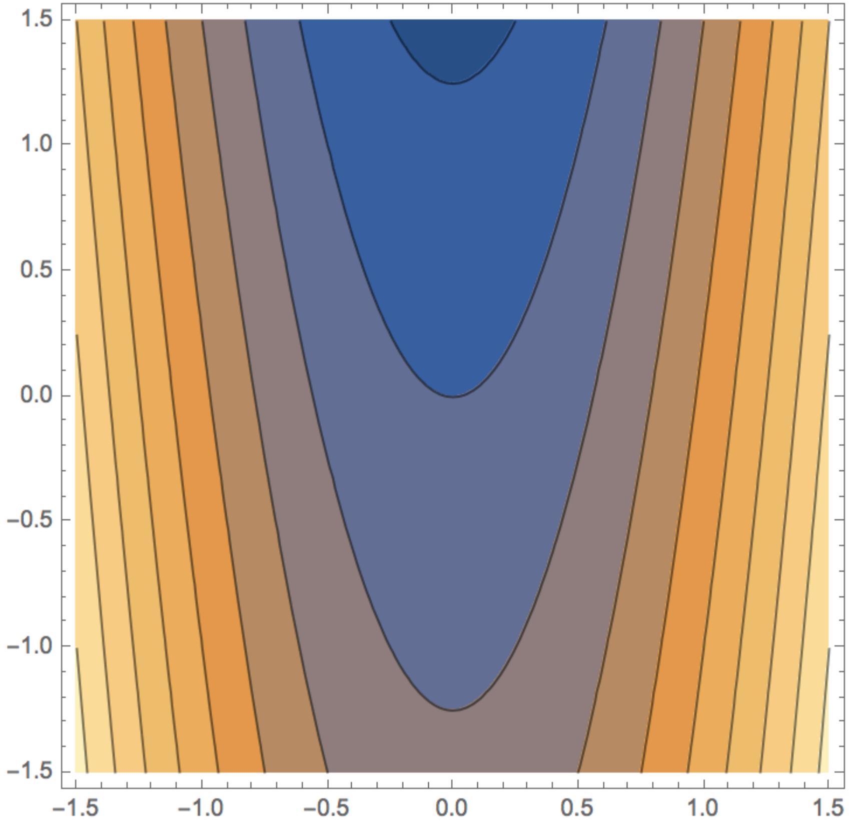
level curves of $f(x,y)$

<u>SOLUTION</u>:
Normals (and derivatives) parallel
$x = 2.5$
$y = 2.5$

$g(x,y) = 0$

Lagrange multipliers example 2



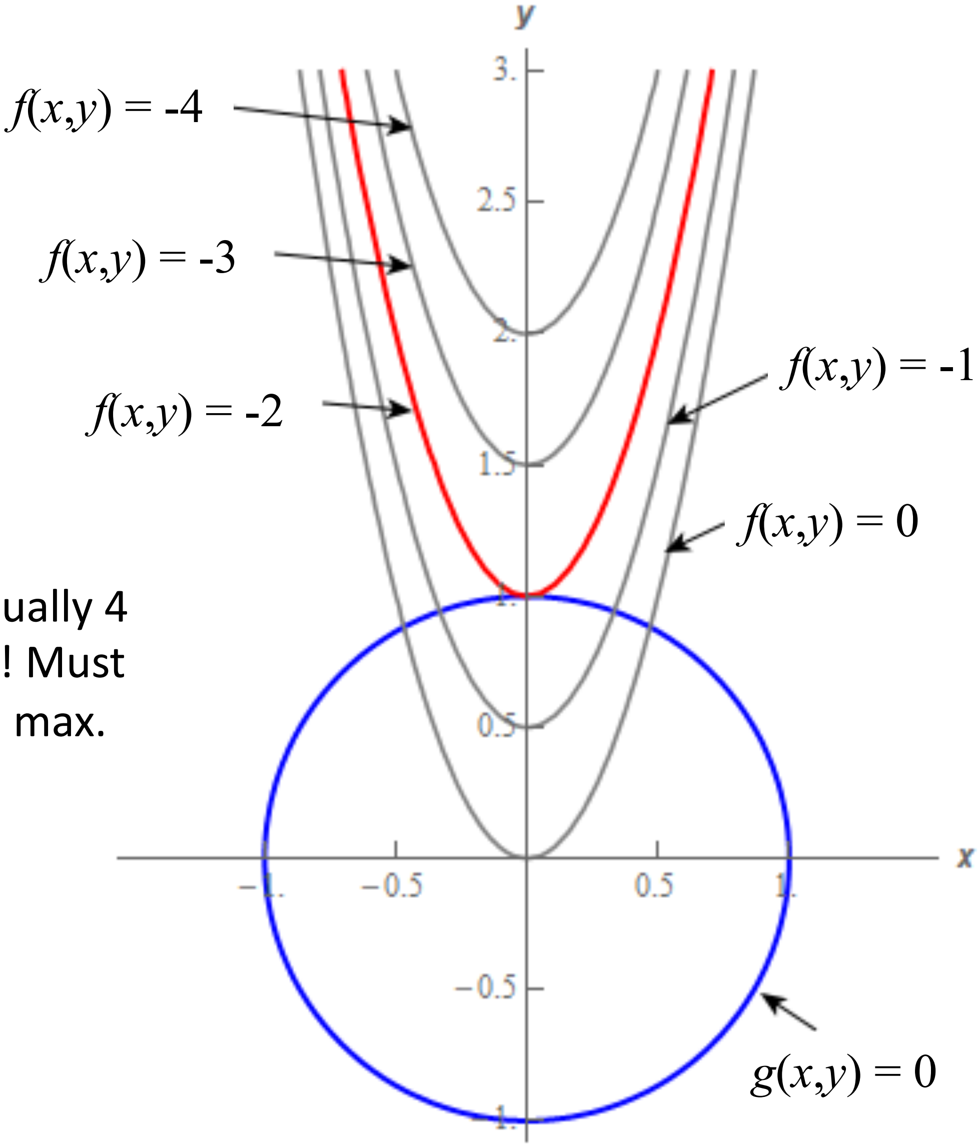


Contour plot of f(x,y)

$$\text{maximize}_{x,y} \quad f(x,y)$$

$$s.t. \quad g(x,y) = 0$$

Lagrange multipliers example 2



Note: there are actually 4 potential solutions! Must plug in to *f* to find max.