Tutorial on Lagrange Multipliers *(find and work with a partner)*

The goal of Lagrange multipliers (for our purposes) is to optimize a function subject to a constraint. With a two dimensional input and one constraint (for example), the problem looks like:

$$\max_{x,y} \quad f(x,y)$$
$$\text{s.t.} \quad g(x,y) = 0$$

The method of Lagrange multipliers allows us to create the Lagrange function (*Lagrangian*):

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

where the scalar λ is called the *Lagrange multiplier*. We will reframe the problem as taking the gradient of $\mathcal{L}$ and setting it equal to the 0 vector. This will give us 3 equations and 3 unknowns (in this case). When we take the derivative of $\mathcal{L}$ with respect to λ and set it equal to 0, this gives us our constraint:

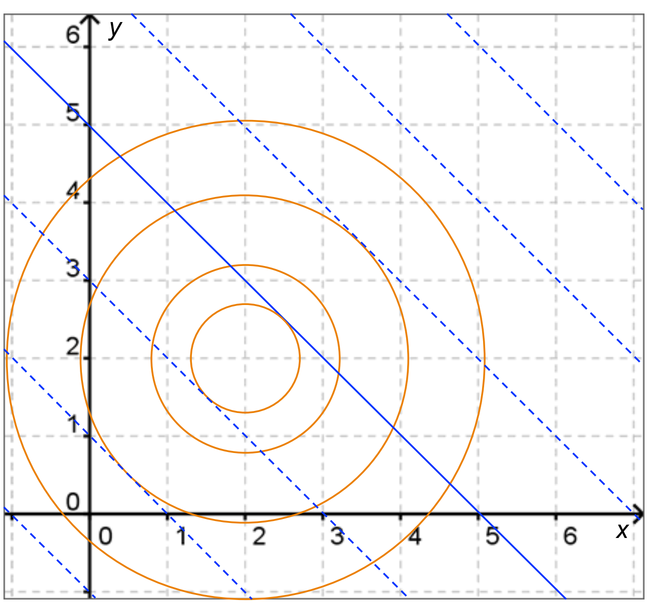
$$\frac{\partial \mathcal{L}(x,y,\lambda)}{\partial \lambda} = g(x,y) = 0$$

And taking the gradient with respect to (x,y) gives us the rest of our equations:

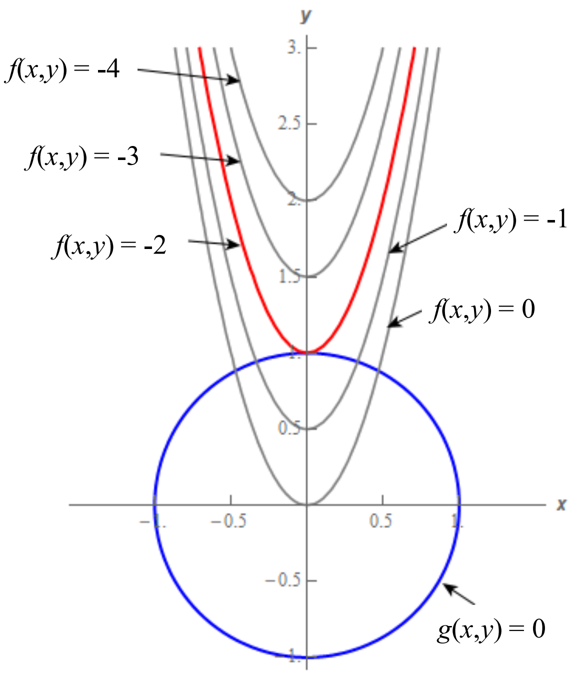
$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

Intuitively – imagine the level curves (contours) of both $f(x,y)$ and $g(x,y)$. Now imagine walking along the contour $g(x,y)=0$. Consider the normal vectors to this curve and the normal vectors to the contours of $f(x,y)$ as this happens. If the normals are *not* parallel, then we could walk further along $g(x,y)=0$ (in some direction) and make the value of f go either up or down. Thus we are not at a maximum. When the normals *are* parallel, we are at a potential min/max of $f(x,y)$. If the normals are parallel, then then derivatives must be parallel, hence the equation above.

1. Let $f(x,y) = 5-(x-2)^2-(y-2)^2$ and $g(x,y) = -5+x+y$. Use the method of Lagrange multipliers to maximize $f(x,y)$, subject to $g(x,y)=0$. On the contour plot below, identify the contour corresponding to $g(x,y)=0$ and draw a few normals to this contour. Show how the normals of f and g are parallel at the solution.



2. Here, let $f(x, y) = 8x^2 - 2y$ and $g(x, y) = x^2 + y^2 - 1$. Again use the method of Lagrange multipliers to maximize $f(x, y)$, subject to $g(x, y) = 0$. This time there are four (correction!) points that satisfy $\nabla \mathcal{L}(x, y, \lambda) = 0$. How can you tell which one is the maximum of f? Contour plot shown below.



Acknowledgements: second example from "Paul's Online Notes" `http://tutorial.math.lamar.edu`